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ON FINITE GROUPS WITH TWO IRREDUCIBLE CHARACTER DEGREES

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ABSTRACT. We characterize non-abelian finite groups with only two irreducible character degree and prime number of non-linear irreducible characters.

1. **Introduction**

Let $\mathrm{Irr}(G)$ be the set of irreducible complex characters of a finite group G and $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$ be the set of irreducible character degrees of G. The question:

Given the set $\mathrm{cd}(G)$, what can be said about the structure of G?

have been studied by several people. In case $\mathrm{cd}(G) = \{1, m\}$, for some integer m, the basic tools for studying the question can be found in Chapter 12 of the well known book of Isaacs [8]. Isaacs proved in Theorem 12.5 and Corollary 12.6 the following results.

Theorem A. *Let G be a finite group and $\mathrm{cd}(G) = \{1, m\}$. Then, at least one of the followings occurs:*
(a) *G has an abelian normal subgroup of index m.*

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(b) *$m = p^e$ for some prime p and G is a direct product of a p-group and an abelian group.*

Theorem B. *Let G be a finite group and $\mathrm{cd}(G) = \{1, m\}$. Then, G' is an abelian group, where G' is the derived subgroup of G.*

If G is non-nilpotent and $\mathrm{cd}(G) = \{1, m\}$, then clearly (a) of Theorem A holds. Recently, Bianchi *et.al.* [4], characterized non-nilpotent groups G with $\mathrm{cd}(G) = \{1, m\}$, and proved the following result.

Theorem C. *Let G be a finite non-nilpotent group. Then, $\mathrm{cd}(G) = \{1, m\}$ if and only if G' is abelian and one of the following holds:*
(a) *m is a prime, and $F(G)$, the fitting subgroup of G, is abelian of index m in G.*
(b) *$G' \cap Z(G) = 1$ and $G/Z(G)$ is a Frobenius group with kernel $\frac{G' \times Z(G)}{Z(G)}$ and a cyclic complement of order $|G : G' \times Z(G)| = m$, where $Z(G)$ denotes the center of G.*

Our notations and terminologies are standard and mainly taken from [8]. We also use the following notations:
Lin(G)= the set of linear characters of G.
$\mathrm{Irr}_1(G) = \mathrm{Irr}(G) \setminus \mathrm{Lin}(G)$.
$ES(c, 2)$= the extra-special group of the order 2^{2c+1}.
(H, N) = the Frobenius group with kernel N and complement H.
C_n= the cyclic group of order n.
$E(q^n)$ = the elementary abelian q-group of rank n.
$Z_i = Z_i(G)$ = the i-th center of G.

Here, we characterize groups G with two irreducible character degrees, that is $\mathrm{cd}(G) = \{1, m\}$, such that the number of irreducible non-linear characters is a prime p. In [1], Berkovich characterized solvable groups for which the degrees of the characters are distinct, except for one pair (having the same degrees). Thus, by [1] we may assume that $p \geq 3$. Firstly, we consider nilpotent groups and prove the following result.

Theorem 1. *Let G be a finite nilpotent group, $p \geq 3$, a prime and m, a positive integer. Then $\mathrm{cd}(G) = \{1, m\}$ and G has p non-linear*

irreducible characters of degree m if and only if one of the followings holds:
(a) $G = C_p \times ES(c, 2)$*; that is, G is the direct product of a cyclic group of order p and an extra-special group of order 2^{2c+1}.*
(b) *G is an special 2-group of order $m^2(p+1)$, and for each subgroup M of index 2 of G', $G/M \simeq ES(c, 2)$, where $2^c = m$.*
(c) *G is dihedral, semidihedral or a generalized quaternion 2-group and $|G| = 4(p+1)$.*

If $\mathrm{cd}(G) = \{1, m\}$, then by Theorem B, G is metabelian. Theorem C characterizes non-nilpotent groups G with $\mathrm{cd}(G) = \{1, m\}$. We complete the characterization of groups G with $\mathrm{cd}(G) = \{1, m\}$ having p non-linear characters of degree m, by considering the non-nilpotent case in the following theorem.

Theorem 2. *Let G be a finite non-nilpotent group, $p \geq 3$, a prime and m, a positive integer. Then $\mathrm{cd}(G) = \{1, m\}$ and G has p non-linear characters of the degree m if and only if G' is abelian and one of the followings holds:*
(a) *$G = (C_m, G')$, where G is a Frobenius group with kernel of order $mp + 1$ and a cyclic complement of order m.*
(b) *$G/Z(G) = (C_m, (G/Z(G))')$, $Z(G) = C_p$ and $G' = E(q^n)$, where $q^n - 1 = m$. Therefore, $G = C_p \times (C_{q^n-1}, E(q^n))$ or $G = C_{p(q^n-1)} \ltimes E(q^n)$; that is, G is a semidirect product with kernel $E(q^n)$.*
(c) *m is a prime, $G' \cap Z_{t-1} = m^{t-2} = p+1$ and $G/Z_{t-1} \simeq (C_m, (G/Z_{t-1})')$, a Frobenius group with kernel of order b, such that $|G| = m^t b$ and $\gcd(m, b) = 1$.*

2. **Proofs**

To prove Theorem 1, firstly we prove the following Lemma for p-groups.

Lemma 1. *Let G be a finite p-group, where p is a prime, $\mathrm{cd}(G) = \{1, m\}$ and $|\mathrm{Irr}_1(G)|$ is odd. Then, at least one of the followings holds:*

(a) *G is an special 2-group, the index of G' is m^2, and for each subgroup M of index 2 of G', $G/M \simeq ES(c,2)$ such that $2^c = m$.*
(b) *G is dihedral, semidihedral or a generalized quaternion 2-group and $|G| = 4(p+1)$.*

Proof. Let $p \neq 2$. Then, by [8, Exercise 3.16], the number of non-linear characters of G is even, which is a contradiction. Thus, $|G| = 2^n$ and $m = 2^c$. Let $|\mathrm{Irr}_1(G)| = x$. From

$$|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2$$

we have

$$2^n = x2^{2c} + |G : G'| = x2^{2c} + 2^k,$$

for some positive integer k. Since x is odd, then $2c = k$ and $|G| = (x+1)2^{2c}$. Hence, $x + 1 = 2^r = |G'|$ and $|G : G'| = 2^{2c}$. Now, we distinguish two cases below.

Case 1: $G' \leq Z(G)$. Since G is nilpotent then $2^{2c} = m^2 = \chi(1)^2$ divides $|G : Z(G)|$, for all $\chi \in \mathrm{Irr}_1(G)$ (see [6, Theorem 8.2]). Since $|G : G'| = 2^{2c}$, it follows that $Z(G) = G'$. Now, let M be an arbitrary normal subgroup of index 2 in G'. Note that M is normal in G, since G' is central. If $|\mathrm{Irr}_1(G/M)| = y$, then

$$|G/M| = 2^{2c+1} = y2^{2c} + 2^{2c}.$$

So $y = 1$ and by [2, Main Theorem], $G/M \simeq ES(c,2)$. Thus, $\Phi(G/M) = (G/M)'$, where $\Phi(G)$ is the Frattini subgroup of G, and $\frac{G/M}{\Phi(G/M)} = \frac{G/M}{G'/M} \simeq G/G'$ is elementary. So, $\Phi(G)G'/G' = \Phi(G/G') = 1$, and hence $\Phi(G) \leq G'$. Therefore, $\Phi(G) = G'$. Now, by [5, Lemma III, 3.14], $G^2 = G' \leq Z(G)$. So for each $g_1, g_2 \in G$, $[g_1, g_2]^2 = [g_1^2, g_2] = 1$, and hence G' is an elementary abelian group. Thus, G is an special 2-group. Therefore, (a) holds.
Case 2: $G' \nleq Z(G)$. Let $c \neq 1$. By [8, Exercise 5.14], each subgroup of index 2^c in G contains G'. Then, each $\chi \in \mathrm{Irr}_1(G)$ can be induced from a linear character of some normal subgroup of G. Thus, G is an nM-group and by [7, Theorem 3], G' is an elementary abelian subgroup.

(Recall that a group is said nM-group if all its irreducible characters are induced from a linear character of some normal subgroup of G.)

If N is a normal subgroup of G such that G/N is a non-abelian group, since $\mathrm{cd}(G/N) = \{1, 2^c\}$, by [6, Theorem 8.2], 2^{2c} divides $|G/N|$. So, from $|G/N| = \sum_{\chi\in \mathrm{Irr}(G/N)} \chi(1)^2$, we have that 2^{2c} divides $t = |G/N : (G/N)'|$. But $t = |G : G'N| \leq |G : G'| = 2^{2c}$, so that $t = 2^{2c}$ and $N \leq G'$. Therefore, $Z(G) \leq G'$ and for $\chi \in \mathrm{Irr}_1(G)$ if $K = \ker(\chi)$, then $K \leq G'$.

Now, suppose that $\bar{G} = \frac{G}{K}$. Since $cd(\bar{G}) = \{1, 2^c\}$, then from $|\bar{G}| = \sum_{\bar{\chi}\in Irr(\bar{G})} \bar{\chi}(1)^2$ we have that $|Irr_1(G)|$ is odd. By a similar argument for the group $\bar{G}$ instead of G, we have $Z(\bar{G}) \leq \bar{G}' = G'/K$. Since $\bar{G}$ is an nM-group for each $\bar{\chi} \in Irr_1(\bar{G})$, then there exists a normal subgroup $\bar{H}$ of $\bar{G}$ of index 2^c and $\bar{\theta} \in \mathrm{Lin}(\bar{H})$ such that $\bar{\chi} = \bar{\theta}^{\bar{G}}$. Let $\ker(\bar{\theta}) \neq 1$. From $Z(\bar{G}) \leq \bar{G}' \leq \bar{H}$, we have $\ker(\bar{\theta}) \cap Z(\bar{G}) \neq 1$, and so $\bigcap_{x\in\bar{G}}(\ker(\bar{\theta}))^x \neq 1$. Thus, $\ker(\bar{\chi}) \neq 1$ (see [8, Lemma 5.11]). But, for each $xK \in \bar{G}$ we have $\bar{\chi}(xK) = \chi(x)$ and thus $\ker(\bar{\chi}) = 1$, which is a contradiction. Therefore, $\ker(\bar{\theta}) = 1$ and $\bar{H}' = \bigcap\{\ker(\lambda) \mid \lambda \in \mathrm{Lin}(\bar{H})\} = 1$. So, $\bar{H}$ is abelian and by [8, Theorem 2.32], $\bar{H}$ is cyclic. Since $\bar{G}' \leq \bar{H}'$, then $\bar{G}'$ is cyclic. Thus, $\bar{G}'$ is a cyclic elementary abelian group and hence $|\bar{G}'| = 2$. Therefore, $Z(\bar{G}) = \bar{G}'$ and $G' = Z(\chi)$, where $Z(\chi) = \{g \in G \mid |\chi(g)| = \chi(1)\}$. So,

$$G' \subseteq \bigcap_{\chi\in Irr_1(G)} Z(\chi) = Z(G),$$

which is a contradiction. Therefore, $c = 1$, $|G/G'| = 4$, and by [5, Chapter III, Theorem 11.9(a)], G is dihedral, semidihedral or generalized quaternion 2-group. So, (b) holds. □

Now, we are ready to prove Theorem 1.

Proof of Theorem 1: Since G is a non-abelian nilpotent group, then $G = Q \times A$, for some non-abelian Sylow q-subgroup Q and a subgroup A of G. Let A be non-abelian. Let $\chi_1 \in \mathrm{Irr}_1(Q)$, $\chi_2 \in \mathrm{Irr}_1(A)$ and $\lambda \in \mathrm{Lin}(A)$. If $\chi_1(1) = x_1$ and $\chi_2(1) = x_2$, then $\chi_1\lambda(1) = x_1$ and $\chi_1\chi_2(1) = x_1x_2$. So, $x_1, x_1x_2 \in \mathrm{cd}(G)$, which is a contradiction. Thus A is abelian.

Now, every irreducible character of G is of the form $\chi\lambda$, where $\chi \in \mathrm{Irr}(Q), \lambda \in \mathrm{Irr}(A)$. Since the number of non-linear characters is equal to p, then either Q has only one non-linear character and A has p linear characters or else $A = 1$.

If Q has only one non-linear character then by [2, Main theorem], $Q \simeq ES(c, 2)$, for some positive integer c. Thus, $G \simeq ES(c, 2) \times C_p$ and (a) holds.

If $A = 1$ then G is a q-group and by by Lemma 1, G satisfies (b) or (c).

Conversely, suppose $G = C_p \times ES(c, 2)$. Since $ES(c, 2)$ has only one non-linear character, then G has p non-linear characters of the same degree. If G satisfies (c), then G has a normal abelian subgroup of index 2, and thus $\mathrm{cd}(G) = \{1, 2\}$ and $|\mathrm{Irr}_1(G)| = p$.

If G is an special 2-group such that $|G : G'| = m^2 = 2^{2c}$, then G' has $2^r - 1$ distinct subgroups of index 2, where $|G'| = 2^r$ for some positive integer r. If M is one of such subgroups, then $G/M \simeq ES(c, 2)$ has only one non-linear character. Then, for $\chi \in \mathrm{Irr}_1(G)$ there exists only one maximal subgroup M_χ of G' such that $M_\chi \subseteq \ker \chi$. Thus, G has at least $2^r - 1$ non-linear characters of degree 2^c and $|G : G'| = 2^{2c}$. Then, from $\sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 = |G|$, we have $\mathrm{cd}(G) = \{1, 2^c\}$ and $|\mathrm{Irr}_1(G)| = p$. □

Proof of Theorem 2: Since G has only p non-linear characters of degree m, then

$$|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 = pm^2 + |G : G'|.$$

Since G' is an abelian normal subgroup G, then by Ito's Theorem, see [8, Theorem 6.15], $\chi(1)$ divides $|G : G'|$. If $|G : G'| = mp^2$, then $|G'| = 2$ and G is nilpotent, which is a contradiction. So, $|G : G'|$ is one of the followings:

(i) $|G : G'| = my$, such that $y \neq p$ and y divides m. If m is not a prime, then by [3, Lemma 2] we have $G = (C_m, G')$. If m is not a prime, then by Theorem C, $G' \cap Z(G) = 1$ and $G/Z(G) = (C_m, \frac{G' \times Z(G)}{Z(G)})$. So,

$|Z(G)| = y$. If $|\mathrm{Irr}_1(G/Z(G))| = x$, then

$$|G/Z(G)| = \frac{pm^2 + my}{y} = xm^2 + \left|\frac{G}{Z(G)} : \frac{G'}{Z(G)}\right| = xm^2 + m.$$

So $y = 1$ and $G = (C_m, G')$ and (a) holds.

(ii) $|G : G'| = pmy$, such that y divides m. We have $G = pm^2 + pmy$. If $y \neq 1$ then $|G'| \leq m$, and by [8, Exercise 5.14], G is nilpotent, which is a contradiction. So, $|G| = pm(m+1)$. Suppose m is a prime. By Theorem A, G has a normal abelian subgroup M of index m. By [8, Theorem 12.12], $|M \cap Z(G)| = p$. If $Z(G) \nleq M$, then $G = Z(G)M$. So, G is abelian, which is a contradiction. So, $Z(G) \leq M$ and $Z(G) = C_p$. Now, suppose m is not a prime. Then, by Theorem C, $G' \cap Z(G) = 1$ and $G/Z(G) = \left(C_m, \frac{G' \times Z(G)}{Z(G)}\right)$. Since $|G| = pm(m+1)$, then $|Z(G)| = p$ and $Z(G) = C_p$. If $Z(G) \leqslant G'$, then $|\frac{G}{Z(G)} : (\frac{G}{Z(G)})'| = pm$. Hence, by the above argument, $m(m+1) = xm^2 + pm$ and $m+1 = xm+p$, which is a contradiction. So, $Z(G) \nleq G'$ and $|\frac{G}{Z(G)} : (\frac{G}{Z(G)})'| = m$. Let x be the number of non-linear characters of $G/Z(G)$. Then,

$$|G/Z(G)| = \sum_{\chi \in \mathrm{Irr}(G/Z(G))} \chi(1)^2 = xm^2 + m,$$

and hence $x = 1$. Therefore, by [2, Main Theorem], $G/Z(G) = (C_{q^n-1}, E(q^n))$ is a Frobenius group with elementary abelian kernel $E(q^n) = (G/Z(G))' \simeq G'$ and cyclic complement C_{q^n-1}, for some prime q and integer n such that $m = q^n - 1$.

Now G' is a minimal normal subgroup of G. Since H is a proper subgroup of G' which is normal in G, then $|\frac{G}{H} : (\frac{G}{H})'| = |G : G'| = mp$. Hence, if G/H, has r non-linear characters of degree m, then $|G/H| = rm^2 + pm$. So, $(rm^2 + pm) \mid mp(m+1) = |G|$ and $(rm+p) \mid m(p-r)$ so $(rm+p) \mid (m+1)$, which is a contradiction. Since G is a non-nilpotent group, then there exists one maximal normal subgroup M of G. Now, G' is a minimal normal abelian subgroup of G, and thus $G' \cap M = 1$, and $G = M \ltimes G'$. Now, $G/Z(G) = (C_{q^n-1}, E(q^n))$ and $G' \leq (C_{q^n-1}, E(q^n))$. Hence, $G = Z(G) \times (C_{q^n-1}, E(q^n))$ or $G = C_{p(q^n-1)} \ltimes E(q^n)$, a semidirect product with kernel $G' = E(q^n)$. So, (b) holds.

(iii) $|G : G'| = m^2$. We have $|G| = m^2(p+1)$. Let m be a prime. Now G has a normal abelian group A of index m and by [8, Lemma 12.2], $|Z(G) \cap A| = m$. If $Z(G) \nleq A$, then G is abelian, and so $Z(G) \leq A$ and $|Z(G)| = m$. Suppose that $|G| = m^t b$, such that $\gcd(m, b) = 1$. From $\sum_{\chi \in \mathrm{Irr}(G/Z(G))} \chi(1)^2 = |G/Z(G)|$, we have $mb = xm^2 + |\frac{G}{Z(G)} : (\frac{G}{Z(G)})'|$, where $|\mathrm{Irr}_1(G/Z(G))| = x$. Thus, m^2 divide $|\frac{G}{Z(G)} : (\frac{G}{Z(G)})'| = |\frac{G}{G'Z(G)}|$. Now, $|G : G'| = m^2$, and so $Z(G) \leq G'$. For $1 \leq i < t$, G/Z_i is a non-nilpotent group and $\mathrm{cd}(G/Z_i) = \{1, m\}$. Thus, by the above argument, $|Z(G/Z_i)| = m$. Hence, $Z_{t-1}(G)$ is a subgroup of order m^{t-1} of G. If $G = G'Z_{t-1}$, then G/Z_{t-1} is abelian, which is a contradiction. Therefore $|G' \cap Z_{t-1}| = m^{t-2}$.

Now, from $\sum_{\chi \in \mathrm{Irr}(G/Z_{t-1})} \chi(1)^2 = |G/Z_{t-1}|$, we have $mb = xm^2 + |\frac{G}{Z_{t-1}} : (\frac{G}{Z_{t-1}})'|$, where $|\mathrm{Irr}_1(G/Z_{t-1})| = x$. Hence, $|\frac{G}{Z_{t-1}} : (\frac{G}{Z_{t-1}})'| = m$. Thus, by [3, Lemma 2], $G/Z_{t-1} = (C_m, (G/Z_{t-1})')$.

Now, let m be not a prime. Then, by Theorem C, $G/Z(G) = (C_m, \frac{G' \times Z(G)}{Z(G)})$, and $|\frac{G' \times Z(G)}{Z(G)}| = p + 1$. Then, $m \mid p$ and $m = p$ is prime, which is a contradiction.

Conversely, if G satisfies (a) or (b), then $Z(G) \cap G' = 1$ and $G/Z(G) = (C_m, \frac{G' \times Z(G)}{Z(G)})$. So, by Theorem C, $\mathrm{cd}(G) = \{1, m\}$ and $|\mathrm{Irr}_1(G)| = p$. If G satisfies (c), then $|G| = m^t b$ and G' is a normal abelian subgroup of index m^2 of G. So, G' contains N, a normal abelian subgroup of order b of G, and thus NZ_{t-1} is a normal abelian subgroup of index m of G. So, by [8, Theorem 6.15], $\mathrm{cd}(G) = \{1, m\}$. Now, since $|G : G'| = m^2$, then $|\mathrm{Irr}_1(G)| = p$. This completes the proof. $\square$

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References

[1] Y. Berkovich, Finte solvable groups in which only two non-linear irreducible characters have equal degree, *J. Algebra* **184** (1996) 584-603.

[2] Y. Berkovich, D. Chillag and M. Herzog, Finite groups in which the degrees of the non-linear irreducible characters are distinct, *Proc. Amer. Math. Soc.* **115** (1992) 955-959.

[3] Y. Berkovich and A. Mann, On sums of degrees of irreducible characters, *J. Algebra* **199** (1998) 646-665.

[4] M. Bianchi, A. G. B. Mauri, M. Herzog, G. Qian and W. Shi, Characterization of non-nilpotent groups with two irreducible character degrees, *J. Algebra* **284** (2005) 326-332.

[5] B. Huppert, *Endliche Gruppen I*, Speringer-Verlag, 1967.

[6] B. Huppert, *Character Theory of Finite Groups*, de Gruyter, Berlin, 1998.

[7] A. Mann, Normally monomial p-groups, *J. Algebra* **300** (2006) 2-9.

[8] I. M. Isaacs, *Character Theory of Finite Groups*, Academic Prees, New York, 1976.

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CONVERGENCE THEOREMS FOR TWO FINITE FAMILIES OF UNIFORMLY L-LIPSCHITZIAN MAPPINGS

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Abstract. Here, we consider a composite iterative process for two finite families of Lipschitzian mappings in a real Banach space. Our results mainly improve and extend the recent results of [S.S. Chang, Some results for asymptotically pseudo-contractive mappings and asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. 129 (2001) 845-853], [Y.J. Cho, J.I. Kang, H. Zhou, Approximating common fixed points of asymptotically nonexpansive mappings, Bull. Korean Math. Soc. 42 (2005) 661-670] and some others.

1. Introduction and Preliminaries

Throughout the paper, we assume that E is a real Banach space, E^* is the dual space of E, C is a nonempty closed convex subset of E and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2,\ \|f\| = \|x\|\}, \quad \forall x \in E,$$

where $\langle \cdot, \cdot\rangle$ denotes the duality pairing between E and E^*. The single-valued normalized duality mapping is denoted by j.

Let $T : C \to C$ be a mapping. Recall the following definitions.

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